2017

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

6 - NOV 1931

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Volume XIX, No. 4.

August 1931.

MADRAS :

PRINTED BY SRINIVASA VARADACHARI & CO.

190, MOUNT ROAD.

Annual Subscription: Rs. 9.] [Single Copy : Rupee One and Annas Eight.

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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

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then it is easy to prove that

$$\phi\left(A, s, \tfrac{1}{2}\right) = \phi\left(\frac{A}{2^{\beta}}, \frac{s}{2^{\alpha}}\right) - \phi(A, s) \qquad \ldots \quad (2.11)$$

For

$$\phi\left(A, s, \tfrac{1}{2}\right) + \phi(A, s) = 2\sum_{n=0}^{\infty} \frac{(-s)^n}{n!}\frac{1}{\beta}\,\Gamma\left(\frac{1+n\alpha}{\beta}\right)\left(Ae^{-\frac{\pi}{2}i}\right)^{-\left(\frac{n\alpha+1}{\beta}\right)}$$

$$+\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{(-s)^n}{n!}\frac{(Ai)^{\nu}}{\nu!}\left\{\zeta\left(-n\alpha-\nu\beta, \tfrac{1}{2}\right) + \zeta(-n\alpha-\nu\beta)\right\}$$

$$= \sum_{n=0}^{\infty} \frac{\left(-\frac{s}{2^{\alpha}}\right)^n}{n!}\frac{1}{\beta}\,\Gamma\left(\frac{1+n\alpha}{\beta}\right)\left(\frac{A}{2^{\beta}}e^{-\pi i/2}\right)^{-(n\alpha+1)/\beta}$$

$$+\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\left(-\frac{s}{2^{\alpha}}\right)^n}{n!}\frac{\left(\frac{Ai}{2^{\beta}}\right)^{\nu}}{\nu!}\,\zeta(-n\alpha-\nu\beta)$$

$$= \phi\left(\frac{A}{2^{\beta}}, \frac{s}{2^{\alpha}}\right).$$

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3. Next, consider the series (1.3). As in § 2, if s is real and positive, we have

$$H(s) = \sum_{n=0}^{\infty} \psi(n+a)\, e^{-s(n+a)^{\alpha}} = \int_0^{\infty} \psi(x)\, e^{-sx^{\alpha}}\, dx$$

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$$+\, i\int_0^{\infty} \frac{\psi(it)\, e^{-s(it)^{\alpha}}}{e^{2\pi ai+2\pi t}-1}\,dt - i\int_0^{\infty} \frac{\psi(-it)\, e^{-s(-it)^{\alpha}}}{e^{-2\pi ai+2\pi t}-1}\,dt$$

$$= I_1 + I_2 + I_3, \text{ say.} \qquad \ldots \quad (3.1)$$

Now suppose $\psi(z) = \sum\limits_{0}^{\infty} c_n z^n$, and let the apparent order of $\psi(z)$ be $\beta\ (< \alpha)$. Then

$$|c_n| < e^{-(1/\beta-\varepsilon)\, n \log n},\text{*} \qquad \ldots \quad (3.2)$$

for every positive ε, and all sufficiently large values of n.

---

* Whittaker and Watson, *loc. cit.*, pp. 267—269.

Therefore

$$\sum_{n=0}^{\infty} | c_n | r^n = O\left(r^{1/2k} . e^{(k/e)} r^{1/k}\right)^*, \qquad \ldots (3.3)$$

(r real and positive), where the constant implied in the symbol O is independent of r, and $k = \frac{1}{\beta} - \varepsilon$. As ε is arbitrary, $\frac{1}{k}$ can be made smaller than α.

Let us now consider I_1.

$$I_1 = \int_0^{\infty} \left(\sum_0^{\infty} c_n x^n\right) e^{-sx^{\alpha}} dx \qquad \ldots (3.4)$$

$$= \sum_{n=0}^{\infty} c_n \int_0^{\infty} x^n e^{-sx^{\alpha}} dx,$$

since

$$\int_0^{\infty} \sum_0^{\infty} | c_n x^n | . | e^{-sx^{\alpha}} | dx$$

is convergent by virtue of (3.3).

Therefore

$$I_1 = \sum_0^{\infty} \frac{c_n}{\alpha} \int_0^{\infty} e^{-sx} x^{\left(\frac{1+n}{\alpha}-1\right)} dx, \qquad \ldots (3.5)$$

$$= \frac{1}{\alpha} \sum_0^{\infty} c_n \frac{\Gamma\left(\frac{1+n}{\alpha}\right)}{s^{(1+n)/\alpha}}.$$

Next, consider I_2.

$$I_2 = i\int_0^{\infty} \frac{dt}{e^{2\pi ai+2\pi t}-1} \left\{ \sum_{n=0}^{\infty} c_n (e^{\pi i/2}\, t)^n \sum_{\nu=0}^{\infty} \frac{(-s)^{\nu}}{\nu!} (te^{\pi i/2})^{\nu\alpha} \right\}, \qquad \ldots (3.6)$$

$$= i\int_0^{\infty} \frac{c_0}{e^{2\pi ai+2\pi t}-1} dt$$

$$+ \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' c_n \frac{(-s)^{\nu}}{\nu!} e^{\pi i(1+n+\nu\alpha)/2} \int_0^{\infty} \frac{t^{(n+\nu\alpha)}\, dt}{e^{2\pi ai+2\pi t}-1},$$

---

* Hardy, '*Orders of Infinity*', pp. 52–53.

the inversion of order of summation and integration being easily justified by using the equality (3.3), and $\sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}'$ meaning as in § 2.

Therefore

$$I_2 = i\int_0^{\infty} \frac{c_0}{e^{2\pi ai+2\pi t}-1}\,dt$$

$$+\sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \sum_{\mu=1}^{\infty} c_n \frac{(-s)^{\nu}}{\nu!} e^{\pi i(1+n+\nu\alpha)/2}$$

$$\times e^{-2\pi\mu ai} \int_0^{\infty} e^{-2\pi\mu t}\, t^{(n+\nu\alpha)}\,dt, \qquad \ldots (3.7)$$

$$= i\int_0^{\infty} \frac{c_0}{e^{2\pi ai+2\pi t}-1}\,dt$$

$$+\sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \sum_{\mu=1}^{\infty} c_n \frac{(-s)^{\nu}}{\nu!}\,\Gamma(1+n+\nu\alpha)\,\frac{e^{\pi i(1+n+\nu\alpha)/2-2\pi\mu ai}}{(2\pi\mu)^{(1+n+\nu\alpha)}},$$

the inversion of order of summation and integration being justified as in § 2.

Similarly,

$$I_3 = -i\int_0^{\infty} \frac{c_0}{e^{-2\pi ai+2\pi t}-1}\,dt$$

$$+\sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \sum_{\mu=1}^{\infty} c_n \frac{(-s)^{\nu}}{\nu!}\,\Gamma(1+n+\nu\alpha).\,\frac{e^{-\pi i(1+n+\nu\alpha)/2+2\pi\mu ai}}{(2\pi\mu)^{(1+n+\nu\alpha)}}.$$

$$\ldots (3.8)$$

Therefore

$$I_2+I_3 = \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' c_n \frac{(-s)^{\nu}}{\nu!}\,\zeta(-n-\nu\alpha,\,a)+c_0\,\zeta(0,\,a)$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} c_n \frac{(-s)^{\nu}}{\nu!}\,\zeta(-n-\nu\alpha,\,a). \qquad \ldots (3.9)$$

Hence

$$H(s) = \sum_{n=0}^{\infty} c_n \left\{ \frac{\Gamma\left(\frac{1+n}{\alpha}\right)}{\alpha\, s^{(1+n)/\alpha}} + \sum_{\nu=0}^{\infty} \frac{(-s)^{\nu}}{\nu!} \zeta(-n-\nu\alpha, a) \right\}. \quad (3.10)$$

This equation has been obtained on the assumption that s is real and positive. But now it can be easily proved that the single series represents an integral function of $\left(s^{-\frac{1}{\alpha}}\right)$, and the double series represents an integral function of s. So that the function H (s) has $s = 0$ for its sole finite singularity, and that if a cut be made in the s-plane along the negative real axis, and if $H_0(s)$ be the function which is defined by continuation in the cut-plane from the Dirichlet's Series, then

$$H_0(s) = \sum_{n=0}^{\infty} c_n \left\{ \frac{1}{\alpha} \Gamma\left(\frac{1+n}{\alpha}\right) \exp.\left(-\frac{1+n}{\alpha} \log s\right) \right.$$

$$\left. + \sum_{\nu=0}^{\infty} \frac{(-s)^{\nu}}{\nu!} \zeta(-n-\nu\alpha, a) \right\}, \quad \ldots(3.11)$$

where log s has its principal value.

If, however, α is the reciprocal of a positive integer (> 1), then H (s) has an essential singularity at the origin.

4. Lastly, consider the series (1.1). It is absolutely and uniformly convergent in the half-plane $\sigma \geqslant 1 + \delta > 1$, and represents there an analytic function of s, say, H (s). Now we proceed to show that H (s) is an integral function of s, and to obtain an expression for it.

Suppose, for a moment, that s is real and > 1. Let A be positive, and suppose $-1 < \alpha < 0$, where $a > -\alpha$. Then if $(z+a)^{\beta}$ and $(z+a)^{-s}$ have their principal values, we have as in § 2,

$$H(s) = \sum_{0}^{\infty} e^{Ai(n+a)^{\beta}} (n+a)^{-s}$$

$$= -\int_{\alpha}^{\alpha+i\infty} \frac{e^{Ai(z+a)^{\beta}} (z+a)^{-s} dz}{e^{2\pi i z} - 1}$$

$$+ \int_{\alpha}^{\alpha-i\infty} \frac{e^{Ai(z+a)^{\beta}} (z+a)^{-s} dz}{e^{2\pi i z} - 1},$$

$$\equiv I_1 + I_2, \text{ say.} \quad \ldots \quad (4.1)$$

Now each of these two integrals can be shown to represent an integral function of s. Hence $H(s)$ is an integral function of s, and the equation (4.1) is true for all values of s.

Now let $a \to -a$, and suppose s to be real and negative, so that the integrands of I_1 and I_2 remain regular at the lower limits. Then

$$I_1 = \int_{-a}^{-a+i\infty} e^{Ai(z+a)^\beta} (z+a)^{-s}\, dz + i \int_0^\infty \frac{e^{Ai(it)^\beta} (it)^{-s}\, dt}{e^{2\pi ai+2\pi t}-1}$$

$$= I'_1 + I''_1 \text{ say} \qquad \ldots (4.2)$$

$$I_2 = -i \int_0^\infty \frac{e^{Ai(-it)^\beta} (-it)^{-s}\, dt}{e^{-2\pi ai+2\pi t}-1}.$$

Treating I'_1 like the integral I_1 of § 2, we can prove that if real and negative

$$I'_1 = e^{i\pi(1-s)/2\beta} \int_0^\infty e^{-At^\beta}\, t^{-s}\, dt, \qquad \ldots (4.3)$$

$$= e^{i\pi(1-s)/2\beta} \frac{1}{\beta} \int_0^\infty e^{At}\, t^{(1-s-\beta)/\beta}\, dt,$$

$$= e^{i\pi(1-s)/2\beta} \frac{1}{\beta} \Gamma\left(\frac{1-s}{\beta}\right) (A)^{(s-1)/\beta},$$

$$= \frac{1}{\beta} \Gamma\left(\frac{1-s}{\beta}\right) \left(Ae^{-i\pi/2}\right)^{(s-1)/\beta}.$$

Next, consider I''_1.

$$I''_1 = \int_0^\infty \left\{ \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} \frac{(Ai)^n}{n!} t^{(n\beta-s)} e^{\pi i (n\beta+1-s)/2 - 2\pi a\mu i - 2\pi\mu t} \right\} dt \qquad \ldots (4.4)$$

$$= \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} \frac{(Ai)^n}{n!} e^{\pi i(n\beta+1-s)/2 - 2\pi\mu ai} \int_0^\infty e^{-2\pi\mu t}\, t^{(n\beta-s)}\, dt,$$

$$= \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} \frac{(Ai)^n}{n!} e^{\pi i(n\beta+1-s)/2 - 2\pi\mu ai} \cdot \frac{\Gamma(1+n\beta-s)}{(2\pi\mu)^{(1+n\beta-s)}},$$

the inversion of order of summation and integration being justified by the fact that

$$\int_0^\infty \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} \left| \frac{(Ai)^n}{n!} \right| . \left| t \right|^{n\beta - s} e^{-2\pi\mu t} \, dt, = \int_0^\infty \frac{e^{|A| t^\beta} . t^{-s}}{e^{2\pi t} - 1} \, dt$$

is convergent, s being real and negative, $\beta \leqslant 1$, and $0 < A < 2\pi$ when $\beta = 1$.

Similarly,

$$I_2 = \sum_{n=0}^{\infty} \sum_{\mu=1}^{\infty} \frac{(Ai)^n}{n!} e^{-\pi i (n\beta + 1 - s)/2 + 2\pi a \mu i} \frac{\Gamma(1 + n\beta - s)}{(2\pi\mu)^{(1+n\beta-s)}} \quad \ldots (4.5)$$

Therefore

$$I_1'' + I_2 = \sum_{n=0}^{\infty} \frac{(Ai)^n}{n!} \frac{2\,\Gamma(1 + n\beta - s)}{(2\pi)^{(1+n\beta-s)}} \left\{ \sin\left(-\frac{\pi}{2}(n\beta - s)\right) \times \right.$$

$$\left. \sum_{\mu=1}^{\infty} \frac{\cos 2\pi\mu a}{\mu^{(1+n\beta-s)}} + \cos\left(-\frac{\pi}{2}(n\beta - s)\right) \sum_{\mu=1}^{\infty} \frac{\sin 2\pi\mu a}{\mu^{(1+n\beta-s)}} \right\}$$

$$= \sum_{n=0}^{\infty} \frac{(Ai)^n}{n!} \zeta(s - n\beta, a), \quad \ldots (4.6)$$

by the formula of Hurwitz already referred to.

Therefore

$$I_1 + I_2 = \frac{1}{\beta} \Gamma\left(\frac{1-s}{\beta}\right) \left(A e^{-\pi i/2}\right)^{(s-1)/\beta}$$

$$+ \sum_{n=0}^{\infty} \frac{(Ai)^n}{n!} \zeta(s - n\beta, a). \quad \ldots (4.7)$$

This equation has been obtained on the assumption that s is real and negative. But now we wish to show that the right side represents an integral function of s, so that the equation is true for all values of s.

Choose any arbitrary positive integer p, and suppose $\sigma < p\beta$. Then the right side of (4.7)

$$= \frac{1}{\beta} \Gamma\left(\frac{1-s}{\beta}\right)\left(A\, e^{-\pi i/2}\right)^{(s-1)/\beta} + \sum_{n=0}^{p-1} \frac{(Ai)^n}{n!} \zeta(s-n\beta, a)$$

$$+ \sum_{n=p}^{\infty} \frac{(Ai)^n}{n!} \zeta(s-n\beta, a).$$

Now all the singularities of the first term neutralize those of the second term, and the two terms taken together represent an analytic function of s in any finite part of the half-plane $\sigma < p\beta$. As regards the infinite series, it can be shown to be uniformly convergent for all bounded values of s in the half-plane $\sigma < \beta p$, by using the formula of Hurwitz for $\zeta(s - n\beta, a)$.

Hence the right side of (4.7) is an analytic function of s in any finite part of the half-plane $\sigma < \beta p$; and as p is arbitrary, it is an integral function of s.

It follows therefore that

$$H(s) = \frac{1}{\beta} \Gamma\left(\frac{1-s}{\beta}\right) \left(A\, e^{-\pi i/2}\right)^{(s-1)/\beta} + \sum_{n=0}^{\infty} \frac{(Ai)^n}{n!} \zeta(s-n\beta, a) \quad \ldots (4.8)$$

This result remains true when $a = 1$.*

Now let

$$H(s, A, a) = \sum_{n=0}^{\infty} e^{Ai\,(n+a)^{\beta}} (n+a)^{-s},$$

then it is easy to prove that

$$H(s, A, \tfrac{1}{2}) + H(s, A, 1) = 2^s H(s, A/2^{\beta}, 1). \quad \ldots (4.9)$$

---

* See my paper already referred to.

For

$$H(s, A, \tfrac{1}{2}) + H(s, A, 1) = \frac{2}{\beta}\,\Gamma\left(\frac{1-s}{\beta}\right)\left(Ae^{-\pi i/2}\right)^{(s-1)/\beta}$$

$$+ \sum_{n=0}^{\infty} \frac{(Ai)^n}{n!}\,\zeta(s-n\beta)\,(2^{s-n\beta}),$$

$$= 2^s\left\{\frac{1}{\beta}\,\Gamma\left(\frac{1-s}{\beta}\right)\left(\frac{A}{2^\beta}\,e^{-\pi i/\beta}\right)^{(s-1)/\beta}\right.$$

$$\left. + \sum_{n=0}^{\infty} \frac{\left(\frac{Ai}{2^\beta}\right)^n}{n!}\,\zeta(s-n\beta)\right\}$$

$$= 2^s\, H\left(s, \frac{A}{2^\beta}, 1\right).$$

Several interesting results can be deduced from the formula (4.9).

5. Now we are going to use the formula (4.8) to obtain an expression for $\zeta(s, a)$, where a is real and $0 < a < 1$, in terms of the zeta-function of Riemann.

$\zeta(s, a)$ is an analytic function of s all over the plane with the exception of a simple pole at $s = 1$. If $\sigma < 0$, then, by the formula of Hurwitz, we have

$$\zeta(s, a) = \frac{2\,\Gamma(1-s)}{(2\pi)^{1-s}} \times$$

$$\left\{ \sin\left(\frac{\pi}{2}s\right) \sum_{n=1}^{\infty} \frac{\cos 2\pi an}{n^{(1-s)}} + \cos\left(\frac{\pi}{2}s\right) \sum_{n=1}^{\infty} \frac{\sin 2\pi an}{n^{(1-s)}} \right\} \quad \ldots (5.1)$$

Also if s be real and $-1 < s < 0$, then, using the formula (4.8) we have

$$\sum_{n=1}^{\infty} e^{2\pi ani}\, n^{-(1-s)} = \Gamma(s)\,(2\pi a)^{-s}\, e^{\pi is/2} + \sum_{n=0}^{\infty} \frac{(2\pi ai)^n}{n!}\,\zeta(1-s-n) \ldots (5.2)$$

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It follows from this that

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi an)}{n^{1-s}} = \Gamma(s)\,(2\pi a)^{-s} \cos\left(\frac{\pi}{2}s\right)$$

$$+ \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi a)^{2n}}{(2n)!}\,\zeta(1-s-2n), \quad \ldots (5.3)$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi an)}{n^{1-s}} = \Gamma(s)\,(2\pi a)^{-s} \sin\left(\frac{\pi}{2}s\right)$$

$$+ \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi a)^{2n+1}}{(2n+1)!}\,\zeta(-s-2n). \quad \ldots (5.4)$$

Hence if s be real and $-1 < s < 0$, we have

$$\zeta(s, a) = \frac{1}{a^s} + \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ \sin(s\pi/2) \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi a)^{2n}}{(2n)!}\,\zeta(1-s-2n) \right.$$

$$\left. + \cos(s\pi/2) \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi a)^{(2n+1)}}{(2n+1)!}\,\zeta(-s-2n) \right\}$$

$$\ldots (5.5)$$

since both sides of this equation are analytic functions of s, save for isolated values of s at which they have poles, this equation, proved when $-1 < s < 0$, persists for all values of s save those isolated values.

6. According to the principle enunciated by me in my paper.* *On two divergent Dirichlet's Series*, the following divergent series may be defined and expressions for their sums obtained :—

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* *Proc. Benares Math. Soc.*, Vols. VII-VIII.

$$\sum_{n=0}^{\infty} e^{Ai\,(n+a)^{\beta}+k\,(n+a)^{\gamma}-s\,(n+a)^{\alpha}}, \qquad 1 \geqslant \beta > \gamma > \alpha > 0,\ k > 0; \qquad \ldots (6.1)$$

$$\sum_{n=0}^{\infty} e^{Ai\,(n+a)^{\beta}+k\,(n+a)^{\gamma}}\,(n+a)^{-s}, \qquad 1 \geqslant \beta > \gamma > 0,\ k > 0. \qquad \ldots (6.2)$$

Lastly, I have to add that it has not yet been possible to obtain expressions similar to those obtained in §§ 2 and 4 for the sums of series such as

$$\sum_{n=0}^{\infty} e^{Ain^{\beta}}(n+a)^{-s}, \quad 0 < \beta < 1; \qquad \ldots (6.3)$$

$$\sum_{n=0}^{\infty} e^{Ai\,(n+a)^{\beta}-sn^{\alpha}}, \quad 0 < \alpha < \beta < 1, \qquad \ldots (6.4)$$

though it is easy to prove that they represent integral functions of s.

LOWER SEGMENTS OF M-CURVES.

By S. Mukhopadhyaya,

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1. Convex M–Curves.

The name M-Curve (Monotropic Curve) is due to Stäckel. H. Mohrmann, in adopting the name, gives the following precise definition :

"A singularity-free (that which does not intersect itself) real branch of an analytic curve, which divides the Euclidean plane into two, and only two, regions, and for which the curvature at every finite point is limited and different from zero, will be called a limited monotropic curve or simply an M-Curve." (*Mathematische Annalen*, **72, 1912**).

In the following investigations the M-Curves considered will not be supposed necessarily analytic. We will not moreover suppose that there exists necessarily a definite circle of curvature anywhere. We will call our M-Curves simply Convex M-curves. They may be characterized as follows :

(i) Every M-Curve defines a convex region in the Euclidean plane and separates it from the rest of the plane. The former region is ***interior*** and the latter region ***exterior*** to the curve.

(ii) There is a positive order of the points of the curve, which is clock-wise from the point of view of an interior point. This positive order will be called a ***continuous*** order as the M-Curve will be supposed to pass continuously between any two successive points of the order.

It does not follow that if any five points are in continuous order on an M-Curve they are necessarily in continuous order on the conic determined by the five points, unless the conic is an ellipse. In Theorem IV we will state sufficient conditions in order that any five points in conti-

nuous order on a segment of an M-Curve may also be in continuous order on the conic determined by them.

(iii) There is a definite tangent at each point of the M-Curve, with a positive sense along it, which is clock-wise from the point of view of an interior point. The tangent turns ***continuously*** to the right as the point moves along the curve in the positive sense.

Every M-Curve falls under one or other of the three following classes: ***closed***, ***semi-closed*** and ***open***, corresponding with their prototypes, the ellipse, the parabola and the single branch of a hyperbola, respectively.

A closed M-Curve is an *oval*. A semi-closed or open M-Curve will be called a *bow*.

2. Lower Segments of M-Curves.

A finite segment M (EF) ≡ S, of an M-Curve will be called a ***lower segment***, if the tangents at E and F meet on the side of EF in which S lies. The lower segment is always a continuous arc. The remainder of the M-Curve will be denoted by S′ and called the ***upper segment***. In the case of an oval the upper segment is also a continuous arc; but in the case of the bow the upper segment consists of two infinite arcs with finite ends E and F, separated from one another.

If in the case of an oval the tangents at E and F be parallel, the oval will be said to be divided into two ***even segments***. An even segment may be considered, indifferently, either as a lower segment or as an upper segment.

The portion of the interior of an M-Curve which lies on the side of EF in which S or S′ lies will be respectively called the interior of S or S′. The exterior of S or S′ is similarly defined.

3. Three-Pointic Circles.

A circle determined by any three points of an M-Curve which are either all distinct or consist of two consecutive points and a third point distinct from them, will be called a ***three-pointic circle*** to the M-Curve.

The diameters of all three-pointic circles to an M-Curve will be supposed to have a finite lower bound d and an upper bound D, which is finite or infinite according as the M-Curve is an oval or a bow.

It may be observed that no three-pointic circle as above defined can have a minimum or maximum diameter. It is easily seen that given any three-pointic circle another exists of a smaller as well as larger radius. (*Vide*: *Tohoku Mathematical Journal*, Vol. 34, (1931) p. 123).

It follows that a circle of diameter equal to or less than d or a circle of diameter equal to or greater than D, where D is finite, cannot be three-pointic to a circle.

If a circle c of diameter $\leqq d$ be drawn to touch an M-Curve at any point P and fall on the same side of the tangent in which the M-Curve lies, then c will fall entirely within the M-Curve, for every three-pointic circle which touches the M-Curve at P will have a greater diameter and consequently lie entirely outside c. For a similar reason, if a circle C of diameter $\geqq D$ be drawn to touch an oval at any point P and fall on the same side of the tangent at P in which the oval lies, then C will fall entirely outside the oval. These results are comparable to those deduced by H. Hurwitz (*Mathematische Annalen*, 57, 1903), for an analytic curve, and also to those given by W. Blaschke (*Kreis und Kugel*, 1916, p. 116)).

The above results will also hold for any finite arc s of an M-Curve, d denoting the lower bound and D the upper bound of all three-pointic circles to s. In this case not only d but also D is finite. The existence condition of a circle of curvature at any point P of an M-Curve may then be stated as follows :

If e denote an arbitrary arc of an M-Curve of which P is an interior point and if $D-d$ tend to vanish with s, we say that a circle of curvature exists at P.

THEOREM I.—*Any chord* $EF \leqq d$ *of an M-Curve divides the M-Curve into a lower segment* S *and an upper segment* S', *such that if* P *be a point on* S *and* P' *on* S', *the angle* EPF *is obtuse and the angle* $EP'F$ *is acute. Also the tangents at* E *and* F *meet at a point* T *on the side of* EF *in which* S *lies, such that each of the angles* TEF *and* TFE *is acute.*

The circle c on EF as diameter cannot touch the M-Curve at E or F, neither can it meet S or S$'$ anywhere, for in either case c would be a three-pointic circle.

Consequently c will either fall inside S or outside S. But c cannot fall inside S, as then S will cease to be a lower segment. Therefore c will fall outside S and inside S$'$. The Theorem follows at once.

Cor.—If EF be a chord of a conic equal to or less than the latus rectum the above theorem will necessarily hold, as the lower bound of the diameters of the three-pointic circles to a conic is the latus rectum. The conic is supposed to be either an ellipse, or a parabola or a single branch of a hyperbola.

THEOREM II.—*Any chord EF equal to $d \sin \theta$, $0 < \theta \leqslant \pi/2$, of an M-Curve, divides the M-Curve into a lower segment S and an upper segment S$'$, such that if P be a point of S and P$'$ of S$'$, the angle EPF is greater than $\pi - \theta$, and the angle EP$'$F is less than θ. Also the tangents at E and F meet at a point T, on the side of EF in which S lies, such that each of the angles TEF and TFE is less than θ.*

By Theorem I, EF divides the M-Curve into a lower segment S and an upper segment S$'$. Draw a circle c of diameter d to pass through E and F and have its centre on the side of EF in which S$'$ lies. EF divides c into a minor segment which falls outside S and a major segment which falls inside S$'$, whence the Theorem follows.

Cor.—The above theorem will hold for any conic if for d we take the latus rectum. The conic is supposed to be either an ellipse, or a parabola or a single branch of a hyperbola.

THEOREM III.—*If EF be a chord of an oval equal to $D \sin \theta$, $0 < \theta < \pi/2$, then EF divides the oval into a lower segment S and an upper segment S$'$, such that if P be a point of S and P$'$ of S$'$, the angle EPF is less than $\pi - \theta$ and the angle EP$'$F is greater than θ. Also the tangents at E and F meet at a point T on the side of EF in which S lies, such that each of the angles TEF and TFE is greater than θ.*

The proof is similar to that of Theorem II. It may be observed that every chord EF of an oval is less than D, as the diameter of a three-pointic circle through E and F and any third point of the oval less than D. The angle θ has a maximum value equal to $\text{Sin}^{-1}\, a/D$, where a is the length of the maximum chord of the oval.

Cor.—The perimeter of the oval is greater than $2D\, \text{Sin}^{-1}\, a/D$, but less than $D\pi$.

4. Five-Pointic Conics.

A five-pointic conic C to an M-Curve may be defined to be a conic which is determined by any five points of the M-Curve no three of which are consecutive. If the conic C be a hyperbola the five points which determine C will lie on the same branch of the hyperbola, as the M-Curve is convex.

The concept of consecutive points on an M-Curve requires some elucidation. As the M-Curve has a definite tangent at each point P, there exist *two* definite consecutive points at P. But our M-Curves are not supposed to possess a definite circle of curvature at P. Hence there do not exist *three* definite consecutive points on the M-Curve at P. Nevertheless if a given circle c touch as well as cross the M-Curve at any point P, three definite consecutive points exist on the M-Curve *along* c. Generally there will exist a class of circles which touch as well as cross the M-Curve at P. Each member of this class will define its own three consecutive points at P. Two of the three consecutive points will be common to all the circles of the class but the third consecutive point will be different for different circles of the class. Hence if of the five points on an M-Curve, three are consecutive, the conic through them will not be determinate.

The latera recta of all five-pointic conics to an M-Curve, we will suppose, have a lower bound l. This lower bound l will be equal to or less than the lower bound of the diameters of all three-pointic circles, as the latus rectum of a conic passing through any five points will be smaller than the diameter of a circle determined by any three of the five points.

THEOREM IV.—*If P (1), P (2), P (3), P (4), P (5) be any five points in continuous order on an M-Curve no three being consecutive, and the segment of the M-Curve between any successive two being a lower segment, and if further each of the chords P (1) P (2), P (2) P (3), P (3) P (4), P (4) P (5) be equal to or less than l, the lower bound of the latera recta of all five-pointic conics, then P (1), P (2), P (3), P (4), P (5) will lie in continuous order on the conic C determined by them.*

If possible suppose the conic C is discontinuous between any two successive points P (1) and P (2).

The chord P (1) P (2) divides the M-Curve into a lower segment S and an upper segment S′, and P (4) is a point in the upper segment as P (3), P (4), P (5) cannot be all consecutive. Hence the angle P (1) P (4) P (2) is acute (Theorem I).

Again P (1) P (2) divides the conic C into a lower segment and an upper segment. Hence the angle P (1) P (4) P (2) is obtuse, as P (4) lies in the lower segment (Cor. Theorem I). This is absurd,

Cor.—If a chord EF equal to or less than l, divide out a lower segment S on the M-Curve, then any five points in continuous order on S will be in continuous order on the conic determined by them.

TAXATION AS AN INSTRUMENT FOR MODIFYING INEQUALITIES IN DISTRIBUTION.

BY M. R. DORESAMIENGAR, M.A., B.SC.,
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It has been claimed that the economist of to-day has prevailed upon the Parliamentary mind and as a result "without interfering with, or abrogating, the essential elements of freedom in the policy of *laissez faire* and without assuming the awkward and dangerous responsibility for fixing prices and incomes, Parliament has interfered materially with the distribution of wealth, has done much to secure a better distribution and has done more probably in the present century to solve the problem of poverty than any previous generation or any other country has been able to do."* Thus of the three agents for bettering the distribution of economic welfare, the employer, the society and the state, prominence has been duly given to the rôle of the last, 'the authoritarian element' in the words of Professor Clay. Naturally it will be desired to ascertain to what extent this authoritarian element has interfered in the economic life of the nation.

The problem is capable of a mathematical enunciation † in the following manner. "If D is the divergence of the actual economic welfare from the ideal one‡ proper to a given distribution of incomes, and

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* *Economic Journal*, 1927, page 16 (Clay; "The authoritarian element in distribution."

† Both in the explicit formulation of this question and in the attempt to furnish an answer I have received considerable help from Messrs, K. B. Madhava and M. Narayana Iyengar of the Mysore University to whom my thanks are due.

‡ *Economic Journal*, 1920, page 349. Dr. Dalton: "The inequality of incomes." When incomes are all equal the corresponding aggregate welfare is taken to be the ideal one.

D′ the divergence proper to the distribution when the income X changes to X′ as a result of taxation only, (the number of persons n at income level X remaining the same at the new level X'), what value does the ratio D′/D assume ?" Obviously this has no dimension in economic welfare or in any other economic quantity. It will be a simple measure of governmental intervention. An estimate of it is of fundamental importance in anticipating the results of a socialistic legislation.

It is necessary at the outset to select some particular functional relation between welfare and income. Of the three functional forms recommended* for use, namely (1) $k \log x$; (2) $k \sqrt{x}$; (3) $k \int e^{-x^2}$ form (2) is selected here†. The second assumption relating to the taxation of incomes is that it is only Edgeworth's scheme‡ that may find favour with the present-day Central or Federal Governments. If $x_1, x_2 \ldots \ldots x_n$ are the actual incomes and $n_1, n_2, \ldots \ldots$ are the numbers of persons at these incomes, the scale of surplus incomes has to be derived on some hypothesis or other. For example the minimum subsistence income can be defined to be the one for which the tax rate according to Edgeworth's plan is zero. In other words if α and β are given constants the minimum income shall be $\alpha^{1/\gamma}$ where $\gamma = 1 - \beta$.

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* *Mysore University Magazine*, March 1928. "Mathematical theory of value" by Prof. K. B. Madhava.

† Form (2) is known as Cramer's function. Of course until experimental evidence with objective measurement is forthcoming one functional form should be considered as good as another If marginal welfare is assumed to decline as the income increases, we can write dw/dx equal to (1) k/x or (considering that the growth of welfare is dependent upon the quantity of it already existing) (2) k/w, or (3) k/x^2. Integrating we get w equal to (1) $k \log x$ (2) $k \sqrt{(x-m)}$ (3) $C - 1/x$. The last is Dr. Dalton's form. But it seems to be just possible that marginal welfare does not decline so rapidly as marginal utility. So a sub-Bernoullian function has been adopted in this paper. Again in these days for well-known reasons incomes should not be glaringly divergent from one another, and the problem of considering values of w for infinite incomes does not arise prominently.

‡ Edgeworth's Tax-formula ; $x - T = \alpha x^{\beta}$.

Therefore $X = x - \alpha^{1/\gamma}$ etc. Had all the incomes been equally distributed there would have been the ideal (and also the maximum) welfare $w_i = k.\ p.\ \sqrt{(\Sigma\ n_1 x_1 / p)}$ where p is the total number of income-holders in the community: the actual aggregate welfare w_a is $k\ (\Sigma\ n_1 x_1^{\frac{1}{2}})$.

Hence $D = w_i - w_a = k p.\ \{ (\Sigma\ n_1 x_1 / p)^{\frac{1}{2}} - \Sigma\ n_1\ x_1^{\frac{1}{2}}/p \}$. As the result of a tax T imposed upon it, let the income x change to ax^{β}. In this event, the ideal welfare w_i proper to the new distribution of incomes is $k.\ p\ (\Sigma\ n_1\ a x_1^{\beta}/p)^{\frac{1}{2}}$ and the actual welfare w'_a is $k \Sigma\ n_1\ (a x_1^{\beta})^{\frac{1}{2}}$. Hence D' is equal to $(w_i' - w'_a)$ and this in turn is equal to

$$kpa^{\frac{1}{2}}\ \{ (\Sigma\ n_1\ x_1^{\beta}/p)^{\frac{1}{2}} - \Sigma\ n_1\ x_1^{\beta/2}/p \},$$

Now $\beta = 1 - \gamma$, $x^{\beta} = x\ (1 - \gamma \log x)$ to a first approximation, powers of $\gamma \log x$ higher than the first being ignored. Therefore D'/D can be taken equal to

$$a^{\frac{1}{2}} \left\{ \frac{\sqrt{\Sigma\ n_1 x_1\ (1 - \gamma \log x_1)/p} - \Sigma\ n_1 x_1^{\frac{1}{2}}\ (1 - \gamma\ \log x)^{\frac{1}{2}}/p}{\left(\Sigma n_1 \frac{x_1}{p}\right)^{\frac{1}{2}} - \Sigma n_1 \frac{x_1^{\frac{1}{2}}}{p}} \right\}$$

Let $A = \Sigma\ n_1 \dfrac{x_1 \log x_1}{\Sigma\ n_1 x_1}$ and $B = \Sigma\ n_1\ \dfrac{x_1^{\frac{1}{2}} \log x_1}{\Sigma\ n_1 x_1^{\frac{1}{2}}}$. Both A and B will lie between $\log x_1$ and $\log x_n$ if $x_1, x_2 \ldots x_n$ are in an ascending order of magnitude. If the difference between the logarithms of the extreme incomes be not considerable, A and B must be nearly the same. Ignoring powers of γA and γB higher than the first it is easy to see that

$$D'/D = a^{\frac{1}{2}} \left\{ \frac{\left(\Sigma n_1 \frac{x_1}{p}\right)^{\frac{1}{2}} \left(1 - \frac{1}{2}\gamma A\right) - \Sigma n_1 \frac{x_1^{\frac{1}{2}}}{p} \left(1 - \frac{1}{2}\gamma B\right)}{\left(\ n_1 \frac{x_1}{p}\right)^{\frac{1}{2}} - \Sigma n_1 \frac{x_1^{\frac{1}{2}}}{p}} \right\}$$

So the ratio D'/D can be taken to be equal to $a^{\frac{1}{2}} \left(1 - \frac{\gamma}{2}.\ \frac{A + B}{2}\right)$.

Thus we can arrive at the conclusion that as a result of Government adopting Edgeworth's scheme of taxation the quantity $w_i - w_a$ which is a measure of the inequalities in welfare shrinks to

$$a^{\frac{1}{2}} \left\{ 1 - \frac{\gamma}{4} \frac{\Sigma\, n_1 x_1 \log x_1}{\Sigma\, n_1\, x_1} + \frac{\Sigma\, n_1 x_1^{\frac{1}{2}} \log x_1}{\Sigma\, n_1\, x_1^{\frac{1}{2}}} \right\}$$

of its original size.

It may be remembered while integrating the tax equation $\frac{dy}{dx} = \frac{k}{x^p}$ where $p = 2 - \beta = 1 + \gamma$ no other economic significance has been given to the constants α and γ excepting that they show that the formula in question is mid-way* between White's and Cassel's formulæ in regard to the effect on progression. In this new analysis they assume another equally important significance: suppose the taxing authorities want to reduce inequalities in welfare by 50 per cent. Clearly α and γ of Edgeworth's plan must be made to satisfy the equation

$$0{\cdot}5 = a^{\frac{1}{2}} \left(1 - \frac{\gamma}{2} \cdot \frac{A + B}{2} \right).$$

Now A and B are quantities depending on the surplus incomes. If we adhere to the definition that the minimum subsistence income is the one for which the tax rate is zero, it can be expected that the general solution of the above equation will be complicated. But to attempt such a general solution seems to be however unnecessary for the subsistence level of incomes for a particular homogeneous community is primarily fixed with reference to the actual economic conditions. Thus A and B are invariable for a given distribution of incomes and a and γ are to be determined from the equation $0.5 = a^{\frac{1}{2}} \left(1 - \frac{\gamma}{2} \cdot \frac{A + B}{2} \right)$. This shows how carefully and neatly a particular scheme of taxation can be brought into being by the legislators to reduce the divergence of welfare to a required extent.

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* *Indian Economic Journal*, Conference Number, 1930, page 327.

The concept of the ratio of divergences of welfare is an important one. In this paper the change in the value of the ratio D′/D as dependent on the instrument of Goverment taxtation has been studied mathematically. But inequalities in wealth and welfare can also be affected in other ways. Thus schemes of insurance, of old age pensions, increased productivity and transfers of wealth determined either by individual whims and fancies or the peculiar inheritance laws of the country are also factors that change the distribution of incomes from time to time. It is to be hoped that in the near future, with improved statistical and mathematical knowledge it would be possible to assess the effect of each one of these factors on the ratio D′/D individually. In the meantime one can profitably study the trend of the divergence* $w_i - w_a$ as a whole, reliable statistics being collected from time to time. It was hoped to collect income statistics of a number of countries for the last fifteen years and trace the trend of the divergence throughout this period. It should have been a really interesting piece of work to trace the values of D (on a relative scale of course) with Russian statistics. The actual statistical work however had to be confined, for various reasons, to the following simple illustration of the central theme on Governmental taxation.

The American Federal incomes of 1925 detailed in the appendix have been selected for purposes of concluding this paper. The choice is clearly appropriate in that it is only with respect to a Federal or Central tax, we can sensibly assume that the object of the tax is to reduce inequalities and not to bestow any immediate and appreciable benefit on the tax-payer. Now the difference between the common logarithms of the extreme incomes being about 1.3, A and B (with respect to base 10) differ from each other by about 0.5 only. As the

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* Divergence D corresponding to some economic period is proportional to

$$p\left(\sqrt{\frac{n_1 x_1}{p}} - \Sigma \frac{n_1 x_1^{\frac{1}{2}}}{p}\right).$$

When the incomes change corrections have to be made for changes in the cost of living index, etc. Only quantities proportional to D, and not the actual values, can be plotted against time.

calculations have shown the divergence D of welfare has been reduced by 5.5 per cent. owing to the imposition of the Federal tax according to Edgeworth's scheme. The values* assumed for α and γ, *viz*, 1.202 and 0.033 have been already predetermined by Edgeworth himself for the American Federal incomes.

A statistical work† of this kind should have received additional importance, if welfare were assumed to be a function of all the incomes and alterations in its functional expression of the type $w(x_1, x_2, \ldots x_n)$ investigated with a few mathematical assumptions of a very general character for both total and marginal welfare. In this way the work would be analogous to that in the theory of capillarity where on the bare assumption that the force of molecular attraction diminishes as the distance between molecules increases and tends to vanish for distances greater than a finite value, known as the 'range of molecular attraction,' important conclusions regarding the tensile strength of liquids and other physical characteristics have been reached. And it is to be earnestly hoped that Governmental and other interventions in 'personal distribution' will be investigated with precision on similar mathematico-economical bases also.

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* Quoted by Prof. Madhava in his "Official Statistics," *Half-yearly Journal of the Mysore University*, Vol. I.

† There is a supplementary work that can be conducted with the given statistics. Thus, if σ and σ' are the standard deviations corresponding to income distributions (x, n) and (a, x^{β}) the ratio σ'/σ is a purely statistical measure of the effect of taxation. Computing σ'/σ to the same degree of approximation as D'/D we can write that

$$\left(\frac{\sigma'}{\sigma}\right)^2 = a^2\left(1 - 2\gamma\frac{A + B'}{2}\right)$$

where A is as defined already and

$$B' = \frac{\Sigma n_1 x_1^2 \log x_1}{\Sigma n_1 x_1^2},$$

A and B' being assumed, as before, to be nearly equal.

APPENDIX.

INDIVIDUAL RETURNS UNDER 25,000 DOLLARS.

(Federal Income-tax 1925).

Col. 1 Incomes	Col. 2 Number of persons	Col. 3 Central values of col. 1	Col. 4 Col. 3 minus 264 dollars	Col. 5 Values of αx^{β}.
Under 1000	98178	...	...	...
1000 to 2000	1071993	1500 dollars	1236	1174
2000 to 3000	842528	2500 ,,	2236	2084
3000 to 4000	747287	3500 ,,	3236	2978
4000 to 5000	580396	4500 ,,	4236	3854
5000 to 6000	174421	5500 ,,	5236	4742
6000 to 7000	121496	6500 ,,	6236	5610
7000 to 8000	89447	7500 ,,	7236	6485
8000 to 9000	65680	8500 ,,	8236	7343
9000 to 10000	52608	9500 ,,	9236	8202
10000 to 11000	39403	10500 ,,	10236	9076
11000 to 12000	31978	11500 ,,	11236	9906
12000 to 13000	26764	12500 ,,	12236	10780
13000 to 14000	22240	13500 ,,	13236	11630
14000 to 15000	18786	14500 ,,	14236	12480
15000 to 20000	62676	17500 ,,	17236	15030
20000 to 25000	34392	22500 ,,	22236	19230

The values of the constants are as follows (all logarithms being taken to base 10):

$$A = 3.70296; \qquad B = 3.54852; \qquad B' = 3.97087.$$

$$\alpha = 1.202, \qquad \beta = 0.967, \qquad \gamma = 0.033$$

Taking $\log_e 10 = 2.30261 = k$,

we find

$$\frac{1}{2}\gamma . k\left(\frac{A+B}{2}\right) = 0.13775.$$

$$\gamma . k . (A + B') = 0.58311.$$

Hence $\sigma'/\sigma = 0.7761,$

and $D'/D = 0.9453$

SOME INEQUALITIES SATISFIED BY THE HYPER-GEOMETRIC SERIES IN TWO VARIABLES.

By Rama Shankar Varma, M. Sc.

1. We know that the Hyper-geometric series* in two variables, defined by the equation

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m, n} \frac{\alpha_{m+n}\, \beta_m\, \beta'_n}{m!\, n!\, \gamma_{m+n}}\, x^m\, y^n,$$

where $\alpha_m = \alpha(\alpha+1)\ldots(\alpha+m-1)$, and $\sum\limits_{m, n}$ means $\sum\limits_{m=0}^{\infty} \sum\limits_{n=0}^{\infty}$,

s expressed by the simple integral†

$$\frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\, F_1(\alpha, \beta, \beta', \gamma, x, y)$$

$$= \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\, du.$$

$$R(\alpha) > 0,\ R(\gamma-\alpha) > 0$$

Now let us suppose that

(1) the variables x and y are positive quantities,

(2) the parameters α, β, β' and γ are real quantities,

and (3) $(\gamma-\alpha) > 1$, in such a way that $\alpha > 1$.

Under these conditions $(1-u)^{\gamma-\alpha-1}$ $(1-ux)^{-\beta}$ $(1-uy)^{-\beta'}$ and $u^{\alpha-1}$ are positive continuous functions of u in the interval $0 \leqslant u \leqslant 1$

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* M. P. Appell : Sur les Fonctions hyper-gèomètriques de plusieurs variables les Polynomes d'Hermite et autres fonctions sphériques dans l'hyperespace, p. 3. In this book, various formulæ concerning Hypergeometric series in two variables used here will be found.

† E. Picard : "Sur une extension aux fonctions de deux variables, du problème de Riemann relatif aux fonctions hypergeometriques," *Comptes rendus*, t. 90 (1880).

THE JOURNAL

OF

THE INDIAN MATHEMATICAL SOCIETY

NOTES AND QUESTIONS.

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N. SUNDARAM IYER, M.A.

Volume XIX, No. 4.

August 1931.

Notes and Questions.

Marginal notes on Sanford's Short History of Mathematics.

In 1930 there appeared "A Short History of Mathematics" by Vera Sanford which is based very largely on the writings of D.E. Smith. Many teachers of our subject will doubtless be especially interested in this work on account of its brevity. Hence it may be desirable to note here a few possible modifications which will enable the teacher to direct the attention to views which differ from those expressed in the text, since the consideration of different views is a more important exercise for the student of the history of mathematics than the collection of opinions which are supposed to be final.

At the bottom of page 3 and the top of the following page it is stated that "the area of a triangle is given as half the product of its base and its side, no stipulation being made as to the size of the included angle." On the contrary Professor W. W. Struve announced recently that the ancient Egyptians used a correct rule to find the area of a triangle. (*Quellen und Studien zur Geschichte der Mathematik*, Quellen, volume I (1930), page 172).

Near the bottom of page 10 it is stated that "Euclid (c. 300 B. C.) has the distinction of being the only man to summarize all the mathematical knowledge of his times, for, contrary to popular impressions, the *Elements* were not concerned with geometry alone." While it is true that the *Elements* of Euclid include also books devoted to arithmetic, it is not true that Euclid summarized therein even all the geometrical knowledge of his day. In particular, the very interesting results found by Hippocrates relating to the area of lunes is not to be found therein. This quotation is misleading also because it stresses an incorrect point of view as regards Euclid's *Elements*.

On page 18 it is stated that "the chief contributions of the Hindus were their system of numerals, the zero, the rules for operations with zero, the study of progressions, and work with quadratics, all of which were borrowed by Western writers." This may be compared with the

statement that "neither the Hindus nor the Arabs made any contribution to our knowledge of arithmetic and geometric progression," (J. Tropfke, *Geschichte der Elementar-Mathematik*, Volume 6 (1924), page 16). It has also not been established that the Hindu knowledge relating to the use of zero was "borrowed by Western writers." It seems now much more likely that it was re-discovered by these writers.

In speaking about Vieta on page 38 it is said that "his works on mathematics, which were published privately for distribution among his friends, show the first use of letters to represent numbers." It is well known that the ancient Greeks had the habit of representing numbers by letters and that this habit appears also in the *Liber Abaci* of Leonardo of Pisa and still more in the works of Jordanus Nemorarius. It is true that Vieta deserves much credit as regards the use of letters for co-efficients, but the use of letters for numbers originated long before his time.

On page 87 it is stated that "in making his computation, Archimedes purposely chose dimensions that erred by excess, and he discussed numbers of the orders that we would denote as multiples of 1, 10^8, 10^{16}, 10^{24}, etc., culminating in 10,000,000 . 10^{56} or 10^{63}". In speaking about the same subject T. L. Heath says that these numbers culminated in a number which is so large that it would require 80,000 million million ciphers in our system to express it, (*History of Greek Mathematics*, volume 2 (1921), page 81). This number is so enormous that 10^{63} is quite insignificant in comparison therewith. (Cf. *Science*, vol. 70 (1929), p. 282).

On page 106 the following statement appears: "Sexagesimal fractions may have been invented by the Babylonians who made use of 60 as a base in their number system. It is certain, at least, that these fractions were used by the Greeks." It is now equally well known that these fractions were used by the Babylonians. Professor O. Neugebauer recently directed attention to the fact that the so-called multiplication tables of the Babylonians were constructed with a view to converting common fractions into sexagesimal fractions, (*Quellen und Studien zur Geschichte der Mathematik*, Studien, volume I (1930), page 193). The sexagesimal fractions seem to have been used by the Babylonians somewhat like decimal fractions are used in modern times.

Near the middle of page 165 it is stated that "equations of the second degree were solved arithmetically by the Egyptians, geometrically

by Euclid and his followers, and algebraically by the Hindus." Recently it has been noted that we now know 19 problems in the Babylonian literature which give rise to quadratic equations, and in 10 of these the method of solution is explicitly given,(*Quellen und Studien zur Geschichte der Mathematik*,—Studien, volume 1 (1930), page 194.) The statement that the Hindus solved the quadratic equation algebraically is somewhat misleading since the Hindus used only numerical co-efficients. It is also not quite exact to say that zero and negative roots were neglected towards the close of the sixteenth century since such roots were considered at this time and even earlier by some of the writers.

About the middle of page 174 there appears the following statement: "The theorem that every algebraic equation has a root leads to the corollary that every algebraic equation of the *n*th degree has *n* roots. This was first established by Gauss (c. 1799), but the Italian writers of the sixteenth century knew the facts for quadratics, cubics, and fourth-degree equations." This seems to imply that the Italian writers of the sixteenth century knew the facts that every algebraic equation of the third degree has three roots and that every such equation of the fourth degree has four roots. This could not have been the case in view of the very limited knowledge of complex numbers which they possessed. In fact, there seems to be no evidence that they even knew definitely that every algebraic quadratic equation has two roots. The fact that the Italian writers of that time had rules for solving the general cubic and the general biquadratic is an entirely different matter since they failed to comprehend the complete significance of these rules.

One of the most wide spread errors relating to the history of mathematics appears on page 197 where it is stated that H. Briggs used the word 'mantissa' to mean the fractional part of a logarithm. This error appears also under the term mantissa in such widely used works of reference as Webster's *New International Dictionary* (1927) and the Century Dictionary (1913) and in many other places. On the contrary, it is now commonly known that no evidence of the use of this term with its modern mathematical meaning has yet been found which is earlier than 1693, more than sixty years after the death of H. Briggs. On the same page of Miss Sanford's History it is also stated erroneously that Jobst Bürgi approached the subject of logarithms through the idea of exponents.

In what precedes attention was called to a few absolute errors which appear in the work under consideration. As an instance of an important relative error relating thereto we may refer to the account of the invention of logarithms, page 191. Whether this account makes a true or a false impression depends upon the definition of the term logarithm which the reader has in mind. According to note number 195 of the *Encyclopedie des Sciences Mathematiques*, tome 1, volume 1, page 59, the precise notion of the logarithm of a number dates only from the instant when the identity of the logarithm of *a* in a system to the base *b*, and that of the exponent to which it is necessary to raise *b* to obtain *a* was recognized. It is only then that the operations of rank three really took place in arithmetic. According to this view the tables of John Napier do not exhibit a precise notion of logarithms since it is universally admitted that they were not constructed with a fixed base in mind. It would be instructive to try to formulate a definition of the term logarithm so as to justify the statement that John Napier constructed the first table of logarithms.

On page 278 it is stated that "the elusiveness of the problem is illustrated by several approaches made to it. Legendre (1752—1833), starting *de novo*, proved that the sum of the angles of a triangle could not be greater than two right angles, but he was unable to prove that it could not be less." This statement seems to contradict the well-known fact that Riemann developed a geometry in which the sum of the angles of a plane triangle is greater than two right angles, for if Legendre could prove that the sum of the angles of a plane triangle could not be greater than two right angles he could obviously prove that Riemann's geometry is impossible. This is contrary also to what appears on page 281 of the history under consideration.

The last sentences on page 306 are as follows: "It should be mentioned, however, that the names co-ordinates, abscissa, and ordinates were contributed by Leibniz (1692); that Newton gave an almost exhaustive treatment of cubics which Plucker completed in 1828, and that the idea of other types of coordinates was not developed until toward the close of the eighteenth century, Fontana, an Italian mathematician, being responsible for the use of polar coordinates." The fact that Leibniz did not contribute the name of abscissa may be seen from the note on this word on the opening page of tome 3, volume 3 of the

Encyclopédie des Sciences Mathematiques, but he seems to have been largely responsible for its general adoption as a mathematical term. From page 39 of this volume it also follows that it is not quite correct to say that Fontana was responsible for the use of polar co-ordinates.

On page 332 Eratosthenes (c. 230 B.C.) is said to have invented a device called the *sieve* for locating prime numbers. Few mathematical developments due to the ancient Greeks are now more widely known or more frequently referred to than this sieve. In 1911 E. Hoppe directed attention in his *Mathematik und Astronomie*, page 284, to the fact that this method seems to have been known to the Greeks long before the time of Eratosthenes, and hence that the common term "sieve of Eratosthenes" is actually a misnomer. Before this time all writers who referred to this subject seem to have credited Eratosthenes with the discovery of this method which is the only one found in the mathematical literature of the ancients for determining all the prime numbers which do not exceed a given number. In fact, most of the writers who referred to this subject since 1911 have also given credit either explicitly or implicitly to Eratosthenes for the discovery of this method. This was, however, not done by J. Tropfke in the third edition of volume 1, 1930, of his *Geschichte der Elementar-Mathematik*, page 127.

The preceding notes aim merely to direct attention to views which differ from those expressed in the text under consideration. It is not implied that these views should be accepted in preference to those of the text without further examination of the evidences relating thereto. The desirability of considering possible definitions of the terms involved so as to make questionable statements to be in accord with the actual facts should perhaps be emphasized since such considerations are apt to exhibit additional facts relating thereto.

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On the Nature of the roots of a cubic and a Biquadratic.

The discussion of the nature of the roots of a cubic or quartic equation given below differs from those usually given in text-books in that it is not based essentially on the equation of squared differences or of Hessian quadratic in the case of a cubic, or of Euler's cubic or reducing cubic in the case of a quartic.

1. Nature of the roots of a cubic equation.

Any general Cubic equation $ax^3 + 3bx^2 + 3cx + d = 0$ with real coefficients may be transformed into

$$z^3 + 3Hz + G = 0$$

where $z = ax + b$, $H = ac - b^2$, $G = a^2d - 3abc + 2b^3$.

This has one real root; let it be α. Suppose

$$z^3 + 3Hz + G = 0 \equiv (z - \alpha)(z^2 + \alpha z + \lambda),$$

so that

$$\lambda - \alpha^2 = 3H, \quad -\alpha\lambda = G.$$

We write also $\alpha^2 = 4\lambda + k$ where k is a real constant.

From these we have:—

$$G^2 = \alpha^2\lambda^2 = 4\lambda^3 + k\lambda^2, \text{ and } \lambda = -H - k/3.$$

$$\therefore \quad G^2 = -4(H + k/3)^3 + k(H + k/3)^2$$

i.e., $$G^2 + 4H^3 = -\frac{k}{27}(9H + k)^2.$$

Since $\alpha^2 = 4\lambda + k$, the quadratic $z^2 + \alpha z + \lambda = 0$ will have two real roots if $k > 0$, two imaginary roots if $k < 0$, two equal roots if $k = 0$.

(i) Therefore the given cubic has *all the three roots real and distinct* if $k > 0$.

Since $G^2 + 4H^3 = -\frac{k}{27}(9H + k)^2$, the necessary and sufficient condition for this is that $G^2 + 4H^3$ must be negative.

(ii) *Suppose the cubic has one real and two imaginary roots.*— In this case k is negative and hence $G^2 + 4H^3 = -k(9H + k)^2/27$

is positive. This condition is both necessary and sufficient. The inequality $G^2 + 4H^3 > 0$ is always true if $H > 0$.

(iii) ***Three real roots of which two are equal.*** If the two roots of the quadratic are equal then $k = 0$. $\therefore$ $G^2 + 4H^3 = 0$ is a necessary condition. This is also sufficient. Because, if $G^2 + 4H^3 = 0$ then either $k = 0$ or $9H + k = 0$; when $k = 0$, the quadratic and hence also the cubic have two equal roots. When $9H + k = 0$, it is easily shown that one root of the quadratic is equal to α.

Therefore the cubic has two roots equal.

(iv) ***All the three roots real and equal.***—Here all the roots must be equal to α,

i.e., $\quad z^2 + \alpha z + \lambda = 0$ has two equal roots, each equal to α.

$\therefore \quad 2\alpha = -\alpha;\ \alpha^2 = \lambda,\quad$ *i.e.*, $\alpha = \lambda = 0,\quad k = 0.$

From the equations $\lambda = -H - k/3$ and $-\alpha\lambda = G$ we get the necessary and sufficient conditions for a cubic to have all its roots equal, namely $H = 0$ and $G = 0$.

2. Nature of the roots of a biquadratic equation.

We write the general biquadratic in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

or

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

where $\quad z = ax + b,\quad H = ac - b^2,\quad I = ae - 4bd + 3c^2$

and

$$G = a^2d - 3abc + 2b^3.$$

Let $z^4 + 6Hz^2 + 4Gz + (a^2I - 3H^2) \equiv (z^2 + 2\lambda z + \mu)(z^2 - 2\lambda z + \mu')$, and let $\lambda^2 = \mu + k = \mu' + k'$. Then we get the following equations

(1) $\lambda^2 = \mu + k$; (2) $\lambda^2 = \mu' + k'$; (3) $\mu + \mu' - 4\lambda^2 = 6H$
(4) $\lambda(\mu' - \mu) = 2G$; (5) $\mu\mu' = a^2I - 3H^2$.

From the equations (1) (2) and (3) we get

(6) $2\lambda^2 = -6H - (k + k')$; (7) $2\mu = -6H - (3k + k')$ and
(8) $2\mu' = -6H - (3k' + k)$.

Eliminating μ and μ' from the equations (3), (4) and (5) we obtain

$$4\lambda^6 + 12H\lambda^4 + \lambda^2(12H^2 - a^2 I) - G^2 = 0, \quad \ldots \quad (9)$$

whence putting $\lambda^2 + H = a^2\phi$,

$$4a^3\phi^3 - I\,a\phi + J = 0. \quad \ldots \quad (10)$$

The equations (9) and (10) are the well-known Euler's cubic and reducing cubic respectively.

Now let us find some suitable expressions for kk' and $k + k'$.

From (1) and (2) $kk' = (\lambda^2 - \mu)(\lambda^2 - \mu')$

$$= \lambda^4 - \lambda^2(6H + 4\lambda^2) + a^2 I - 3H^2$$

$$= a^2 I - 3(\lambda^2 + H)^2 = a^2 I - 3a^4\phi^2. \quad (11)$$

From (10) $a^2\phi^2(4a^2\phi^2 - I)^2 = J^2$ and from (11) $a^2\phi^2 = \dfrac{a^2 I - kk'}{3a^2}$

On substituting for ϕ^2 and simplifying we obtain

$$I^3 - 27J^2 = \frac{kk'}{a^2}\left\{3I - \frac{4kk'}{a^2}\right\}^2. \quad \ldots \quad (12)$$

Similarly from the equations (5), (7) and (8) we obtain on substituting for μ and μ' and simplifying

$$4(a^2 I - 12H^2) = 4(k + k')(k + k' + 6H) - (k - k')^2 \ldots \quad (13)$$

This is an equation giving $(k + k')$. Lastly from (6)

$$k + k' + 6H < 0. \quad \ldots \quad (14)$$

For the detailed discussion of the nature of the roots of a general quartic the results (12), (13) and (14) are enough.

Let us take the different cases one by one.

Case 1.—If all the roots are real, $k > 0$ and $k' > 0$,

i.e., $kk' > 0, \; k + k' > 0.$

Hence the necessary conditions from (12), (13) and (14) are

$$I^3 - 27J^2 > 0; \; a^2 I - 12H^2 < 0 \text{ and } H < 0.$$

These conditions are also sufficient, because if $I^3 - 27J^2 > 0$, from (12) we derive that k and k' must have the same sign. From (13) since

$$a^2 I - 12H^2 < 0, \; 4(k+k')(k+k'+6H) < (k-k')^2,$$

i.e., $$4(k+k')^2 + 24H(k+k') < (k-k')^2. \quad \ldots \quad (15)$$

This inequality becomes absurd if k and k' are negative ; because $4(k+k')^2 > (k-k')^2$ and the other term $24H(k+k') > 0$ Therefore the only alternative is that k and k' must be positive. The inequality (15) holds for all positive values of k and k'.

$\therefore$ The necessary and sufficient conditions that all the roots of a quartic may be real are

$$I^3 - 27J^2 > 0, \; a^2 I - 12H^2 < 0, \; H < 0.$$

Case 2.—If all the roots are imaginary $k < 0$ and $k' < 0$,

i.e., $$kk' > 0 \text{ and } k+k' < 0.$$

The necessary conditions are $I^3 - 27J^2 > 0$ from (12); $(a^2 I - 12H^2)$ may be positive or negative; from (14) H may be greater than or less than zero. Therefore we can say that whenever $I^3 - 27J^2 > 0$ all the roots may be real or all imaginary. But besides this condition if

$$a^2 I - 12H^2 < 0 \text{ and } H < 0$$

are satisfied simultaneously all the roots are real. Out of the three conditions if the first is satisfied and any one of the remaining two is not satisfied, the equation has all its roots imaginary.

Case 3.—If two roots are real and two imaginary, $k > 0$ and $k' < 0$ or *vice versa i.e.*, $kk' < 0$. The necessary condition from (12) is $I^3 - 27J^2 < 0$. Conversely if this inequality is satisfied $kk' < 0$; hence one of the numbers k and k' must be negative and the other positive. Therefore, there are two real roots and two imaginary roots.

Next let us take the cases of equal roots.

Case 4.—All the roots are real; but two roots are equal. For this $k = 0$, and $k' > 0$.

$$\therefore \quad I^3 - 27J^2 = 0, \; a^2I - 12H^2 < 0, \text{ and } H < 0.$$

These conditions are sufficient also. Because if $I^3 - 27J^2 = 0$, from (12) $k = 0$ or $k' = 0$ or both k and k' are equal to zero or

$$3I - \frac{4kk'}{a^2} = 0.$$

When either k or $k' = 0$, it is clear it must have two roots equal and the other two must be real on account of the remaining conditions. Both k and k' cannot vanish since in that case $a^2I - 12H^2 = 0$.

Suppose $$3I - \frac{4kk'}{a^2} = 0; \; k \neq 0, \; k' \neq 0:$$

i.e., $$3a^2I = 4kk' = (k+k')^2 - (k - k')^2.$$

From (5) $$3a^2I = 9H^2 + 3\mu\mu'.$$

$$\therefore \quad \tfrac{3}{4}\{(\mu + \mu')^2 - (\mu - \mu')^2\} + 9H^2$$

$$= \left(\frac{\mu + \mu'}{2} + 3H\right)^2 - (\mu - \mu')^2.$$

whence we obtain on using

$$3H = \frac{\mu + \mu'}{2} - 2\lambda^2$$

and simplifying $$-8\lambda^2(\mu + \mu') = (\mu - \mu')^2.$$

This is exactly the necessary and sufficient condition that the quadratics $z^2 + 2\lambda z + \mu = 0$ and $z^2 - 2\lambda z + \mu' = 0$ might have a root in common. The given quartic will anyhow have two roots equal.

Case 5.—Two equal roots and two imaginary roots. The condition $I^3 - 27J^2 = 0$ ensures that there must be at least two equal roots. If there are two imaginary roots $k' < 0$.

$$\therefore \quad I^3 - 27J^2 = 0; \; a^2I - 12H^2 \nless 0,$$

Case 6.—Three equal roots. Here the conditions for two equal roots must be satisfied.

$$\therefore \qquad I^3 - 27J^2 = 0.$$

Suppose the quadratic $z^2 - 2\lambda z + \mu' = 0$ has two roots equal, *i.e.*, each root is λ. This must be one of the roots of the other quadratic

$$z^2 + 2\lambda z + \mu = 0, \qquad i.e., \quad 3\lambda^2 + \mu = 0,$$

where $\lambda^2 = \mu + k$; also $6H + k = -2\lambda^2$.

$$\therefore \qquad \mu = -3\lambda^2 = \lambda^2 - k. \qquad \therefore \quad k = 4\lambda^2,$$

$$i.e., \qquad 6H + 4\lambda^2 + 2\lambda^2 = 0 \text{ or } \lambda^2 + H = 0.$$

But $(\lambda^2 + H)$ is a root of the reducing cubic and it is zero. Hence the constant term J must vanish

$$\therefore \qquad I = 0 \text{ since } I^3 - 27J^2 = 0.$$

$\therefore$ $I = 0$ and $J = 0$ are the conditions for three roots being equal.

Case 7.—Two pairs of equal roots. In this case

$$k = 0 \text{ and } k' = 0.$$

$$\therefore \qquad I^3 - 27J^2 = 0$$

also

$$a^2I - 12H^2 = 0 \text{ and } H < 0$$

These conditions are not only necessary but also sufficient as may be easily verified.

Case 8.—All the roots equal.

$$k = 0,\ k' = 0,\ I^3 - 27J^2 = 0 \text{ and } a^2I - 12H^2 = 0$$

and also

$$-\lambda = \lambda = \mu = \mu' = 0$$

$$\therefore \qquad H = 0,\ I = 0,\ J = 0$$

The conditions are $H = 0$, $I = 0$ and $J = 0$.

Malleswaram, Bangalore. } M. N. NARASIMHA IYENGAR.

A Theorem on Apolar Quadric Loci.

I. The object of this note is to prove the following useful theorem on apolar quadric loci, which are given by *real* equations.

If Q_1 and Q_2 are non-conical quadric loci in space of $n-1$ dimensions, given by real equations, and if Q_2 is outpolar to Q_1, then there exist real *self-polar simplexes of Q_1 inscribed in Q_2, provided Q_1 as a locus is entirely imaginary (i e., contains no real points).* ... (A)

To prove this theorem let us use real homogeneous co-ordinates $x_1, x_2, \dots x_n$ defined by a real base simplex. Since Q_1, Q_2 are given by real equations, they can be reduced by real linear transformation of co-ordinates to a form involving only the squares of the new co-ordinates. Since by hypothesis Q_1 does not contain real points, it follows that the reduced forms can be taken as:

$$Q_1: \quad X_1^2 + X_2^2 + \dots + X_n^2 = 0$$

$$Q_2: \quad a_1 X_1^2 + a_2 X_2^2 + \dots + a_n X_n^2 = 0,$$

where $a_1, a_2, \dots a_n$ are real. Since Q_2 is outpolar to Q_1,

$$a_1 + a_2 + \dots + a_n = 0.$$

Hence $a_1, a_2, \dots a_n$ cannot all be of the same sign. Hence Q_2 contains real points. We have thus obtained the theorem:

If a quadric locus $Q_2(x) = 0$ is outpolar to a quadric locus $Q_1(x) = 0$, where Q_2 is a real form, and Q_1 a real definite form, then $Q_2(x)$ must be indefinite, and therefore the locus Q_2 contains real points. ... (B)

Now let p be a real point on Q_2 and let P be the polar prime of p with respect to Q_1. Let further S_2, S_1 be the intersections of P with Q_2, Q_1. Since $Q_1(x)$ is a real form, and p is real, it follows that P is real, and that its intersections S_2, S_1 with Q_2, Q_1 are given by real

quadratic forms $S_2(x)$, $S_1(x)$ in $n-1$ variables. Also S_1 cannot contain real points, since Q_1 does not, so that $S_1(x)$ is a definite form. Now the outpolarity of Q_2 to Q_1 implies the existence of real or imaginary self-polar simplexes of S_1, inscribed in S_2, *i.e.*, it implies that S_2 is outpolar to S_1. Thus S_2, S_1 satisfy the conditions of Theorem (B), so that it follows that S_2 must contain real points.

If q is a real point on S_2, it is clear that the above process can be continued, and will give us finally a real self-polar simplex of Q_1 inscribed in Q_2. The theorem is thus proved. A corresponding theorem holds of course for the case in which Q_2 is inpolar to Q_1.

It will be noticed that the proof also shews that the manifold of real self-polar simplexes of Q_1, inscribed in Q_2, has the maximum conceivable dimension. For, an indefinite quadric locus in space of $n-1$ dimensions necessarily contains ∞^{n-2} real points. Thus p can be chosen in ∞^{n-2} ways, q in ∞^{n-3} ways and so on, so that the total number of the simplexes is ∞^{R}, where

$$R = n-2+n-3+\ldots+1 = \frac{(n-1)(n-2)}{2}$$

II. We may use the theorem proved to obtain a type of canonical reduction of two real quadratic forms, of which one is definite, say positive definite.

Let $Q_1(x)$, $Q_2(x)$ be two real quadratic forms of rank n, in n variables $x_1, x_2, \ldots x_n$, Q_1 being positive definite. Then we can determine a real parameter λ so that $Q_1(x) + \lambda Q_2(x)$ is outpolar to $Q_1(x)$. Then by our theorem there exists a real self-polar simplex of the quadric Q_1, inscribed in the quadric $Q_1 + \lambda Q_2$. Taking this as the simplex of reference, $Q_1(x)$ can be reduced to $X_1^2 + X_2^2 + \ldots + X_n^2$, while $Q_1 + \lambda Q_2$ being a quadric circumscribed to the simplex of reference must have the shape $2\Sigma p_{rs} X_r X_s$. Hence Q_2 reduces to the form

$$k \Sigma X^2 + 2 \Sigma q_{rs} X_r X_s.$$

We have thus the result :

Two real quadratic forms Q_1, Q_2, of non-vanishing determinants, of which Q_1 is positive definite, can be reduced by real linear transformation to the forms :

$$Q_1 : \quad X_1^2 + X_2^2 + \dots X_n^2$$

$$Q_2 : \quad k\,(X_1^2 + \dots + X_n^2) + 2\,\Sigma\, q_{rs} X_r X_s.$$

As examples of this result, we have

(1) Let $ax^2 + 2hxy + by^2 = 1$ be the equation of an ellipse in rectangular Cartesian co-ordinates.

Take $Q_1 \equiv ax^2 + 2hxy + by^2$, $Q_2 \equiv x^2 + y^2$. It follows then, that by real homogeneous linear transformation of x, y, we can reduce Q_1 to $X^2 + Y^2$, and Q_2 to $k\,(X^2 + Y^2 + 2XY \cos \omega)$. Now Q_2 is positive definite, and therefore k must be positive. Hence by a further transformation of the form $X = \lambda X'$, $Y = \lambda Y'$, we can reduce Q_1, Q_2 to the forms

$$Q_1 = k'\,(X'^2 + Y'^2)$$

$$Q_2 = X'^2 + Y'^2 + 2X'Y' \cos \omega.$$

The equation $Q_1 = 1$, is evidently the Cartesian equation of the ellipse referred to its equiconjugate diameters. Thus our theorem asserts the existence of a pair of real equiconjugate diameters. For the case of two positive definite forms Q_1, Q_2 in n variables, the theorem would similarly amount to the existence of an infinity of sets of real equal conjugate diameters of the ellipsoid $Q_1 = 1$ in n dimensions.

(2) Consider next the application of the theorem to the case in which $Q_1(X) = X_1^2 + \dots + X_n^2$ is positive definite, while nothing is known about the definiteness of Q_2. We can then by the theorem reduce Q_2 to the shape $k\,(X_1^2 + X_2^2 + \dots + X_n^2) + 2q_{rs}X_rX_s$. If we take $Q_1(X)$ to be the square of the length of the vector $(X_1, X_2, \dots, X_n)$ in n dimensions, it follows at once, that the arbitrary quadric locus $Q_2(X) = 1$ (where Q_2 is a real form) admits sets of n mutually perpendicular diameters of equal length $\sqrt{1/k}$.

R. VAIDYANATHASWAMY.

Reviews.

ELEMENTARY CO-ORDINATE GEOMETRY, by M. L. CHANDRATREYA, *Professor of Mathematics, Deccan College, Poona*; with a Foreword by G. S. MAHAJANI, M.A., Ph.D. 1931.

This is an excellent introduction to the subject written in a careful manner, to serve the needs of the Intermediate and B.A. classes. The examples are carefully chosen so as to bear on the methods of the text, and not to be too difficult. The book closes with a short chapter on the historical development of Analytical Geometry by Dr. Mahajani. We are sure that this will prove a very useful text-book to those for whom it is intended.

R. V.

PLANE TRIGONOMETRY, by B. B. BAGI, *Dharwar*, Tutorial Press, Bombay. 1931.

This is a companion volume to the author's 'Higher Algebra and Trigonometry' published in 1928. This is designed to meet the requirements of students preparing for the Intermediate Examination of Indian Universities. It is well written. Emphasis has rightly been laid on the use of signs in relation to segments of lengths, angles and areas. The chapter on orthogonal projections leading up to the derivation of the formulae for Compound angles is a distinct feature of this book. The chapter on Graphs might have contained the graphs of $a \cos x + b \sin x$ and their applications to the solution of Trigonometric Equations. The Chapter on Heights and Distances ought to contain a treatment of 'Bearing' and 'Points of the Compass' with numerical examples enabling the student to use the mathematical tables freely. The book contains a large variety of examples both solved and unsolved. The printing and the general get up of the book is good.

S. M.

Solutions.

Question 1475.

(M. Mohd. Din Gharib):—Prove that the sum of the squares on one pair of opposite sides of a quadrilateral together with twice the square on the line joining their middle points is equal to the sum of the squares on the other pair of opposite sides with twice the square on the line joining their middle points, without joining the point of intersection of the bisecting lines to its corners.

Solution by Budh Ram Aggarwal, S. Mahadevan, N. P. Subramaniam and Mohd. Din Gharib.

Let E, F, G, H be the middle points of the sides AB, BC, CD, DA of the quadrilateral ABCD.

We have

$$2(BF^2 + AF^2) = 4(EB^2 + EF^2)$$

$$2(FD^2 + FC^2) = 4(GF^2 + GD^2).$$

Adding we obtain

$$BC^2 + 2(AF^2 + DF^2) = (AB^2 + CD^2) + 4(EF^2 + FG^2) \quad \ldots \quad (1)$$

Again

$$2(FA^2 + FD^2) = AD^2 + 4HF^2 \quad \ldots \quad (2)$$

$$4(FE^2 + FG^2) = 2GE^2 + 2HF^2. \quad \ldots \quad (3)$$

Substituting for the left-hand sides of (2) and (3) in (1), we obtain

$$BC^2 + AD^2 + 2HF^2$$

$$= AB^2 + CD^2 + 2GE^2.$$

Question 1497.

(B. B. BAGI):—(1) Through the vertices of a triangle ABC lines AP, BQ, CR are drawn equal and parallel to the sides EF, FD DE of another triangle DEF. Show that the triangles ABC and PQR have the same centroid.

(2) Pairs of points PP′, QQ′, RR′ are taken on the sides BC, CA, AB of a triangle ABC, such that PP′, QQ′, RR′ are proportional to BC, CA and AB. Show that the triangles PQR and P′Q′R′ have the same centroid.

Enunciate corresponding theorems for a polygon.

Solution by N. P. Subramanian.

(1) Let G, G′ be the centroids of the Δs ABC and PQR and let a, b, c and p, q, r be the position vectors of the points A, B, C and P, Q, R. Then

$$p + q + r = a + b + c + \overline{AP} + \overline{BQ} + \overline{CR}$$

$$= a + b + c + \overline{EF} + \overline{FD} + \overline{DE}$$

$$= a + b + c \qquad \because \overline{EF} + \overline{FD} + \overline{DE} = 0.$$

$$\therefore \overline{OG} = \frac{a + b + c}{3} = \frac{p + q + r}{3} = \overline{OG'}$$

$\therefore$ G and G′ coincide.

Therefore, Δs ABC and PQR have the same centroid.

REMARK.—It is not necessary that AP, BQ and CR be equal to EF, FD, DE. It is enough if they are proportional to EF, FD, DE. For in this case also $\overline{AP} + \overline{BQ} + \overline{CR} = 0$ being equal to

$$k(\overline{EF} + \overline{FD} + \overline{DE}). \quad \therefore \ p + q + r = a + b + c$$

and therefore the centroids coincide.

(1) Let G, G′ be the centroids of the Δs PQR, P′Q′R′. Then denoting the position vectors of the points P, Q, R and P′, Q′, R′ by the corresponding small letters, we have

$$p' + q' + r' = p + q + r + \overline{PP'} + \overline{QR'} + \overline{RR'}.$$

But $$\overline{PP'} + \overline{QQ'} + \overline{RR'} = k(\overline{BC} + \overline{CA} + \overline{AB}) = 0.$$

$$\therefore \ p' + q' + r' = p + q + r.$$

Therefore as before the centroids coincide.

Corresponding theorems for a polygon :—

(1) If through the vertices of a polygon $A_1A_2 \ldots\ldots A_n$ lines A_1P_1, $A_2P_2 \ldots\ldots A_nP_n$ are drawn equal and parallel to the sides B_1B_2, $B_2B_3 \ldots\ldots B_nB_1$ of another polygon $B_1B_2 \ldots\ldots B_n$ then the polygons (A) and (P) have the same centroid.

$$\left[\text{For } \Sigma p = \Sigma a + \Sigma \overline{A_1P_1} = \Sigma a + \Sigma \overline{B_1B_2} = \Sigma a.\right.$$

$$\left.\therefore \quad \frac{\Sigma p}{n} = \frac{\Sigma a}{n}\right].$$

(2) Pairs of points P_1Q_1, $P_2Q_2 \ldots\ldots P_nQ_n$ are taken on the sides A_1A_2, $A_2A_3 \ldots\ldots A_nA_1$ of a polygon (A) such that P_1Q_1, $P_2Q_2 \ldots\ldots P_nQ_n$ are proportional to $A_1A_2 \ldots\ldots A_nA_1$. Then the polygons (P) and (Q) have the same centroid.

$$[\text{For } \Sigma p = \Sigma q \text{ since } \Sigma \overline{P_1Q_1} = k \Sigma \overline{A_1A_2} = 0].$$

Also solved by A. A. Krishnaswamy Iyengar, M. V. Seshadri and T. S. Venkataraman.

Question 1504.

(T. TOTADRI IYENGAR) :—If $f(x, y) = 0$ represents a rational algebraic curve which transforms to the polar form $F(r, \theta) = 0$, show that the conditions for a double point are

$$F = 0, \frac{\partial F}{\partial r} = 0, \frac{\partial F}{\partial \theta} = 0.$$

Further show that the point is a node, cusp or conjugate point according as

$$\left(\frac{\partial^2 F}{\partial r \partial \theta}\right)^2 \gtrless \frac{\partial^2 F}{\partial r^2} \cdot \frac{\partial^2 F}{\partial \theta^2},$$

and that the branches cut orthogonally if

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = 0.$$

Solutions by Budhram Aggarwal, M. V. Seshadri, B. Ramamurthy, G V. Krishnaswamy and S. P. Ranganadachar.

If $f(x, y) = F(r, \theta)$, we have

$$\frac{\partial F}{\partial \theta} = -\frac{\partial f}{\partial x} \cdot r \sin\theta + \frac{\partial f}{\partial y} \cdot r \cos\theta \qquad \ldots (1)$$

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x} \cdot \cos\theta + \frac{\partial f}{\partial y} \cdot \sin\theta \qquad \ldots (2)$$

Hence if $f = 0$, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, then

$F = \frac{\partial F}{\partial r} = \frac{\partial F}{\partial \theta} = 0$ (conditions for a double point).

From (1) and (2), it is easy to deduce that if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, then

$$\frac{\partial^2 F}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} \cdot r^2 \sin^2\theta + \frac{\partial^2 f}{\partial y^2} \cdot r^2 \cos^2\theta - 2r^2 \sin\theta \cos\theta \frac{\partial^2 f}{\partial x \partial y} \qquad \ldots (3)$$

$$\frac{\partial^2 F}{\partial r \partial \theta} = -\frac{\partial^2 f}{\partial x^2} \cdot r \sin\theta \cos\theta + \frac{\partial^2 f}{\partial y^2} \cdot r \sin\theta \cos\theta + \frac{\partial^2 f}{\partial x \partial y} r (\cos^2\theta - \sin^2\theta) \qquad \ldots (4)$$

$$\frac{\partial^2 F}{\partial r^2} = \frac{\partial^2 f}{\partial x^2} \cdot \cos^2\theta + \frac{\partial^2 f}{\partial y^2} \sin^2\theta + 2 \sin\theta \cos\theta \frac{\partial^2 f}{\partial x \partial y} \qquad \ldots (5)$$

From (3), (4) and (5) it follows that if we have a double point and

$$\left(\frac{\partial^2 F}{\partial r \partial \theta}\right)^2 \gtreqless \frac{\partial^2 F}{\partial r^2} \cdot \frac{\partial^2 F}{\partial \theta^2}, \text{ then } \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \gtreqless \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}$$

and if

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} = 0, \text{ then } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Hence the results.

Question 1508.

(N. DURAIRAJAN) :—Show that the condition that four points A, B, C, D may be co-normal points on some conic through them is

$$\sum_{1}^{3} a^2 (R_2^2 - R_3^2)(RR_1 + R_2R_3)^2 = 0$$

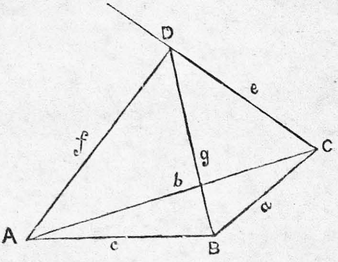
where a, b, c are the lengths of the sides of the triangle ABC and R, R_1, R_2, R_3 are the circum-radii of the triangles ABC, DBC, DCA, DAB respectively.

Solutions by Hukam Chand and K. V. Vedantham.

Let A, B, C, D be co-normal points on the ellipse

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\mu^2} = 1,$$

and α, β, γ, δ respectively their eccentric angles.



If a_1, b_1, etc., denote the lengths of the semi-diameters parallel to BC, CA, etc., it is easy to see that

$$a_1^2 + f_1^2 = c_1^2 + e_1^2 = b_1^2 + g_1^2 = \lambda^2 + \mu^2 \qquad \ldots \quad \text{I}$$

$$a^2 = 4a_1^2 \sin^2 \frac{\beta - \gamma}{2}, \text{ etc.}$$

Also
$$\frac{a_1^2 b_1^2 c_1^2}{\lambda^2 \mu^2} = \frac{a^2 b^2 c^2}{16\Delta^2} = R^2$$

(Δ denoting the area of the triangle ABC. Vide Salmon's ***Algebre***, p. 39) with similar expressions for R_1, R_2, R_3. ... II

The normals at α, β, γ are concurrent if

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin 2\alpha \\ \sin\beta & \cos\beta & \sin 2\beta \\ \sin\gamma & \cos\gamma & \sin 2\gamma \end{vmatrix} = 0$$

i.e., if
$$\begin{vmatrix} \cos\overline{\beta-\gamma} & \cos\overline{\gamma-\alpha} & \cos\overline{\alpha-\beta} \\ \cos\overline{\beta+\gamma} & \cos\overline{\gamma+\alpha} & \cos\overline{\alpha+\beta} \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$= \frac{1}{\lambda^2-\mu^2}\begin{vmatrix} 1-\cos\overline{\beta-\gamma} & 1-\cos\overline{\gamma-\alpha} & 1-\cos\overline{\alpha-\beta} \\ \lambda^2+\mu^2-(\lambda^2-\mu^2)\cos\overline{\beta+\gamma} & \dots & \dots \\ 1 & 1 & 1 \end{vmatrix}$$

i.e., if
$$0 = \begin{vmatrix} \sin^2\frac{\beta-\gamma}{2} & \sin^2\frac{\gamma-\alpha}{2} & \sin^2\frac{\alpha-\beta}{2} \\ a_1^2 & b_1^2 & c_1^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{(\lambda^2+\mu^2)^3}\begin{vmatrix} \sin^2\frac{\beta-\gamma}{2}(a_1^2+f_1^2)^2 & \sin^2\frac{\gamma-\alpha}{2}(b_1^2+g_1^2)^2 & \sin^2\frac{\alpha-\beta}{2}(c_1^2+e_1^2)^2 \\ a_1^2 & b_1^2 & c_1^2 \\ f_1^2 & g_1^2 & e_1^2 \end{vmatrix}$$

$$\therefore\ \Sigma \sin^2\frac{\beta-\gamma}{2}(b_1^2e_1^2-c_1^2g_1^2)(a_1^2+f_1^2)^2 = 0.$$

Multiplying by $\dfrac{a_1^2b_1^2c_1^2g_1^2f_1^2e_1^2}{\lambda^6\mu^6}$, we get

$$\sum a_1^2 \sin^2\frac{\beta-\gamma}{2}\left(\frac{b_1^2f_1^2e_1^2}{\lambda^2\mu^2} - \frac{c_1^2g_1^2f_1^2}{\lambda^2\mu^2}\right) \times$$
$$\left(\frac{a_1b_1c_1}{\lambda\mu}\cdot\frac{a_1e_1g_1}{\lambda\mu} + \frac{b_1e_1f_1}{\lambda\mu}\cdot\frac{f_1c_1g_1}{\lambda\mu}\right)^2 = 0$$

which gives

$$\Sigma\, a^2 (R_2^2 - R_3^2)(RR_1 + R_2R_3)^2 = 0.$$

Question 1510.

(B. B. BAGI):—The points $A_1, A_2, \dots A_n$ represent the complex numbers $\alpha_1, \alpha_2, \dots \alpha_n$, and $B_1, B_2, \dots B_{n-1}$, the $n-1$ roots of the equation

$$\frac{\lambda_1}{x-\alpha_1}+\frac{\lambda_2}{x-\alpha_2}+\dots+\frac{\lambda_n}{x-\alpha_n}=0$$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are real. Show that

$$\frac{A_1 B_1 . A_1 B_2 \dots A_1 B_{n-1}}{A_1 A_2 . A_1 A_3 \dots A_1 A_n}+\frac{A_2 B_1 . A_2 B_2 \dots A_2 B_{n-1}}{A_2 A_1 . A_2 A_3 \dots A_2 A_n}$$
$$+ \dots \text{ to } n \text{ terms} = 1.$$

Solution by B. Ramamurthi, M. V. Seshadri and S. Rajanarayanan.

Let $\beta_1, \beta_2, \dots \beta_{n-1}$ be the roots of the given equation, so that

$$\lambda_1(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)+\lambda_2(x-\alpha_1)(x-\alpha_3)\dots(x-\alpha_n)$$
$$+\dots \text{ to } n \text{ terms} \equiv (\lambda_1+\lambda_2+\dots+\lambda_n)(x-\beta_1)(x-\beta_2)\dots(x-\beta_{n-1})$$

Put here α_1 for x; we have

$$\lambda_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n)$$
$$=(\lambda_1+\lambda_2+\dots+\lambda_n)(\alpha_1-\beta_1)(\alpha_1-\beta_2)\dots(\alpha_1-\beta_{n-1})$$

or $$\frac{(\beta_1-\alpha_1)(\beta_2-\alpha_1)\dots(\beta_{n-1}-\alpha_1)}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\dots(\alpha_n-\alpha_1)}=\frac{\lambda_1}{\lambda_1+\lambda_2+\dots+\lambda_n}.$$

But $$\beta_1-\alpha_1=A_1 B_1;$$

$$\beta_2-\alpha_1=A_1 B_2; \dots \beta_{n-1}-\alpha_1=A_1 B_{n-1}$$

and $$\alpha_2-\alpha_1=A_1 A_2, \text{ etc.}$$

$$\therefore \quad \frac{A_1 B_1 \dots A_1 B_{n-1}}{A_1 A_2 \dots A_1 A_n}=\frac{\lambda_1}{\lambda_1+\lambda_2+ . +\lambda_n}$$

$\therefore$ the *l.h.s.* of the given equality $=\dfrac{\Sigma \lambda_1}{\lambda_1+\lambda_2+\dots+\lambda_n}=1.$

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1596. (V. RAMASWAMI AIYAR, M.A.):—Considering a system of conics having four-pointic contact with one another at a point P :

(1) Prove that the squares of the areas of the ellipses of the system vary as the cubes of the distances of their centres from P.

(2) Frame a like theorem regarding the hyperbolas of the system.

(3) On that ellipse of the system whose eccentricity is least P is an extremity of one of the equi-conjugate diameters.

(4) If C be the centre of this ellipse and C′ the centre of the rectangular hyperbola of the system, CC′ is bisected at P.

(5) The directrix of the parabola of the system bisects PC′ perpendicularly.

(6) The director circles of all the conics of the system cut the circle described on PC′ as diameter orthogonally.

(7) If X, X′ be points dividing CC′ harmonically and if they lie on the same side of P as C, then, they are centres of two *ellipses* of the system of *equal eccentricity*.

(8) Describe the like property in the case where X, X′ (dividing CC′ harmonically) lie on *the other side of* P.

(9) The axes of the conics of the system envelope a parabola whose focus is the reflexion of C′ in the tangent at P to the conics and whose directrix is the line of their centres.

(10) If θ is the angle betwe·n the tangent at P and the line of centres; and PC $= c$ and PX $= x$; and α, β are the semi-axes of the conic of the system of centre X; show that

$$\left(\frac{x}{c} + \frac{c}{x} + 2\right) = \sin^2 \theta \div \left(\frac{\alpha}{\beta^2} + \frac{\beta^2}{\alpha^2} + 2\right).$$

1597. (R. RAMANUJAN):—Given four straight lines measuring 4, 9, 12 and 21 units, solve geometrically the following equations :—

$$\frac{xy(x+y+8)}{21.(x+y)} - \frac{9(x+y)}{25(x+y-8)} = 0;$$

$$(x^2 - y^2)^2 - 64(2x^2 + 2y^2 - 91) = 0.$$

1598. (R. RAMANUJAN):—In a pentagon ABCDE, BE and BD cut AC at F and G respectively, such that A lies on CF produced and E lies on BF produced. Given that BG = 25783, GC = 169153, CD = 263744, DG = 237961, GF = 36271 and $\angle$AED = $\angle$EFD, prove that AC = BE.

1599. (T. R. RAGHAVA SASTRI):—Given a triangle ABC, four conics can be drawn passing through its vertices and having their focus at a point P.

If e_1, e_2, e_3, e_4 are their eccentricities show that

(i) $\frac{1}{e_1^2} + \frac{1}{e_2^2} + \frac{1}{e_3^2} + \frac{1}{e_4^2}$ will be a minimum if P is the symmedian point of the triangle ABC.

(ii) if $\frac{1}{e_1} + \frac{1}{e_2^2} + \frac{1}{e_3^2} + \frac{1}{e_4^2}$ is a given constant, the locus of P will be a circle having its centre at the symmedian point.

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during the months of June and July 1931.

1 American Journal of M[illegible]ematics, **53**, 3.
2 American Mathematical Monthly, **38**, 5.
3 Annals of Mathematics, **32**, 3.
4 Astrophysical Journal, **73**, 4.
5 Bulletin of the American Mathematical Society, **37**, 4, 5, 6.
6 Bulletin of the Calcutta Mathematical Society, **23**, 1.
7 Bulletin des Sciences Mathematiques, **55**, April and May.
8 Japanese Journal of Mathematics, **7**, 4, **8**, 1.
9 Mathematical Gazette, **15**, 214 (2 copies).
10 Mathematische Annalen, **104**, 5.
11 Monthly Notices of the Royal Astronomical Society, **91**, 5, 6.
12 Nieuw Archief Voor Wiskunde, Vol. **17**.
13 Philosophical Magazine, **11**, 73, 74.
14 Proceedings of the Royal Society of London, **131**, 818.
15 Proceedings of the Cambridge Philosophical Society, **27**, 2.
16 Proceedings of the Physico-Mathematical Society of Japan, **13**, 3, 4, 5, 6,
17 Quarterly Journal of Mathematics, **2**, 6.
18 Revista Mathematica Hispano Americana, **6**, 3, 4.
19 Rendiconti del Seminario Matematico, Anno. II, N. 1.
20 Universidad Nacional De La Plata, No. 92.

Books and Reprints.

1 Geometry of Four Dimension, by A. R. Forsyth (Purchased)
2 Theorems Stated by Ramanujan, by G. N. Watson :
(viii) Theorems on Divergent Series.
3—7 (ix) Two continued Fractions.
(xi), (xii) and (xiv) A singular modulus.
8—9 Theorems Stated by Ramanujan (x) and (xiii), by C. T. Preece.
10 Ramanujan's Note-book, by G. N. Watson.
11 Theory of simple continued fractions, by D. M. Mehta.
12 Publications de la Faculte des Sciences, 1 contribution A La Theorie des Series Trigonometriques Generalisees et des Series, a Fonctions orthogonales.
13 Nekolik Poznamek o Markoovoych Retezich.

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